Least Squares Finite Element Method for a Nonlinear Stokes Problem in Glaciology

Max Gunzburger, Lee Eun Jung, Irene Monnesland

Department of CSE, Yonsei University, Seoul Korea

ABSTRACT

Ice is considered to be a slowly moving, viscous and non-Newtonian incompressible fluid. It is therefore commonly modeled by the Stokes equations.
The mathematical model is

$$-\nabla \cdot \left(\alpha \left| \frac{1}{4} (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) \right|^{q-2} (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \,) \right) + \nabla p = \rho \mathbf{g} \ \text{ in } \Omega$$

$$\nabla \cdot \boldsymbol{u} = 0 \ \text{ in } \Omega,$$

where the unknowns are the ice velocity , $\boldsymbol{u} = (\mathrm{u}(x,y,z), \mathrm{v}(x,y,z), \mathrm{w}(x,y,z))$ and the pressure p. The ice density is denoted by ρ , $\mathbf{g} = (0,0,-\mathrm{g})$ is the gravitational acceleration and α is a function in $L^\infty(\Omega)$. The viscosity $\mu = \alpha \left| \frac{1}{4} (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) \right|^{q-2}$ with $q \in (1,2]$, depends on the velocity which makes the equations nonlinear. The problem is supplemented by appropriate boundary conditions.
The purpose of this work is to show the unique existence of a weak solution to the above second order PDE and compute an approximation, by using the least squares finite element method.
To do so, we first linearize the system of equations through a Picard method and solve the system at each iteration with the known function $\mu(\boldsymbol{u}^n)$.
We then rewrite the equations as a first order system and define a corresponding least squares functional. The unique existence of a minimizer to our functional guarantees the unique existence of a weak solution to our problem. We prove that the functional is norm equivalent to a product of appropriately chosen Hilbert space norms. This makes it possible to apply the Lax Milgram theorem, and therefore guarantees the unique existence of the minimizer of the functional.
We finally use the computational advantages of the LSFEM to compute an approximation to the weak solution.

REFERENCES

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