4. Zero-Stability and Convergence for Initial Value Problem

Consider a discrete system

$$A^h E^h = -\tau^h \tag{1}$$

induced from

$$\begin{aligned} AU = F \;\Rightarrow\; & AU = F \;\text{ and }\; AU^h = F + \tau, \;\text{ where } \tau \text{ is LTE} \\ \Rightarrow\; & A(U - U^h) = -\tau \;\;\Rightarrow\;\; AE = -\tau. \end{aligned}$$

- **Stability**

  Solving (1) means

  $$E^h = -(A^h)^{-1}\tau^h.$$

  Taking norms gives

  $$\|E^h\| = \|(A^h)^{-1}\tau^h\| \le \|(A^h)^{-1}\| \; \|\tau^h\|.$$

  If $\|\tau^h\| = O(h^p), p > 0$, we hope that the same will be true of $\|E^h\|$. For this to be true, we need $\|(A^h)^{-1}\|$ needs to be bounded independent to h, as $h \to 0$ so that $\|E^h\|$ goes to zero at least as fast as $\|\tau^h\|$.

  Suppose a finite difference method for a linear DE gives a sequence of matrix equations of the form $A^h U^h = F^h$, where h is the mesh size. We say that the method is stable if $(A^h)^{-1}$ exists for all h sufficiently small and if there is a constant C, independent of h, such that

  $$\|(A^h)^{-1}\| \;\le\; C \quad \text{for all } h \text{ sufficiently small.} \tag{2}$$

- **Consistency**

  We say a method is consistent with the DE if

  $$\|\tau^h\| \longrightarrow 0 \quad \text{as} \;\; h \to 0. \tag{3}$$

- **Convergence**

  $$\text{consistency} + \text{stability} \Longrightarrow \text{convergence}$$

4.1 Stability of one-step method

Consider an IVP

$$u'(t) = f(u(t)) \quad \text{for } t > t_0 = 0$$

with initial data

$$u(0) = \eta.$$

We assume that f is Lipschitz continuous.

- Forward Euler method

$$U^{n+1} = U^n + \Delta t f(U^n) \tag{4}$$

The truncation error is defined by

$$\begin{aligned} \tau^h &= \frac{1}{\Delta t}(u(t_{n+1}) - u(t_n)) - f(u(t_n)) \\ &= \frac{1}{2}\Delta t u''(t_n) + O(\Delta t^2). \end{aligned}$$

Thus, from the above, we have

$$u(t_{n+1}) = u(t_n) + \Delta t f(u(t_n)) + \Delta t \tau^n \tag{5}$$

and then subtracting (5) from (4) yields

$$E^{n+1} = E^n + \Delta t(f(U^n) - f(u(t_n))) - \Delta t \tau^n.$$

By the Lipschitz continuity of f, we have

$$|f(U^n) - f(u(t_n))| \le L|U^n - u(t_n)| = L|E^n|.$$

Therefore,

$$|E^{n+1}| \le |E^n| + \Delta t L|E^n| + \Delta t|\tau^n| = (1 + \Delta t L)|E^n| + \Delta t|\tau^n|$$

and by the induction

$$|E^n| \le (1 + \Delta t L)^n |E^0| + \Delta t \sum_{m=1}^{n} (1 + \Delta t L)^{n-m} |\tau^{m-1}|.$$

$E^0 = 0$ and some calculation provide

$$|1 + \Delta t \lambda| \le e^{\Delta t|\lambda|} \Rightarrow (1 + \Delta t \lambda)^{n-m} \le e^{(n-m)\Delta t|\lambda|} \le e^{n\Delta t|\lambda|} \le e^{|\lambda|T},$$
$$\Delta t \textstyle\sum_{m=1}^{n} |\tau^{m-1}| \le \Delta t n \|\tau\|_\infty \le T\|\tau\|_\infty.$$

Hence

$$|E^n| \le e^{LT} T \|\tau\|_\infty = O(\Delta t) \quad \text{as } \Delta t \to 0$$

proving the method converges.

- General one-step methods
  A general explicit one-step method takes the form

$$U^{n+1} = U^n + \Delta t \Phi(U^n, t_n, \Delta t),$$

for some function Φ which depends on f of course. Assume that $\Phi(u, t, k)$ is continuous in t and Δt and Lipschitz continuous in u.

[Example] For the two-stage Runge-Kutta method, we have

$$\Phi(u, t, \Delta t) = f\left(u + \frac{1}{2}\Delta t f(u)\right).$$

If f is Lipschitz continuous with Lipschitz constant L, then Φ has Lipschitz constant $L' = L + \frac{1}{2}\Delta t L^2$.....

4.2 Zero-stability

The global error can be bounded in terms of the sum of all the on-step errors and hence has the same asymptotic behavior as LTE as $\Delta t \to 0$. This form of stability is often called zero-stability in ODE theory, to distinguish it from other forms of stability that are of equal importance in practice. The fact that a method is zero-stable (and converges as $\Delta t \to 0$) is no guarantee that it will give reasonable results on the particular grid with $\Delta t > 0$ that we want to use in practice. Other stability issues of a different nature will be taken up in the following chapters.

5. Absolute Stability for Ordinary Differential Equations

[Example] Consider an ODE

$$u'(t) = -2100(u - \cos t) - \sin t \quad \text{with I.C. } u(0) = 1.$$

The solution for the above ODE is $u(t) = \cos t$(we can find the solution by using Duhamel's principle) and LTE is $O(\Delta t)$. But now if we compute with Forward Euler and step size $\Delta t = 10^{-3}$, we obtain $U^{2000} = -0.2453 \times 10^{77}$ with an error of magnitude 10^{77}. The computation behaves in an "unstable" manner, with an error that grows exponentially in time. Since the method is zero-stable and $f(u,t)$ is Lipschitz continuous in u (with Lipschitz constant $L = 2100$), we know that the method is convergent, and indeed with sufficiently small time steps we achieve very good results. In this case, something dramatic happens between $\Delta t = 0.000976$ and $\Delta t = 0.000952$. For smaller values of Δt we get very good results, whereas for larger values of Δt there is no accuracy whatsoever.

5.1 Absolute stability

To determine whether a numerical method will produce reasonable results with a given value of $\Delta t > 0$, we need a notion of stability that is different from zero-stability. The one that is most basic and suggests itself from the above example is absolute stability.

First, consider the simplest case :

$$u'(t) = \lambda u(t). \tag{6}$$

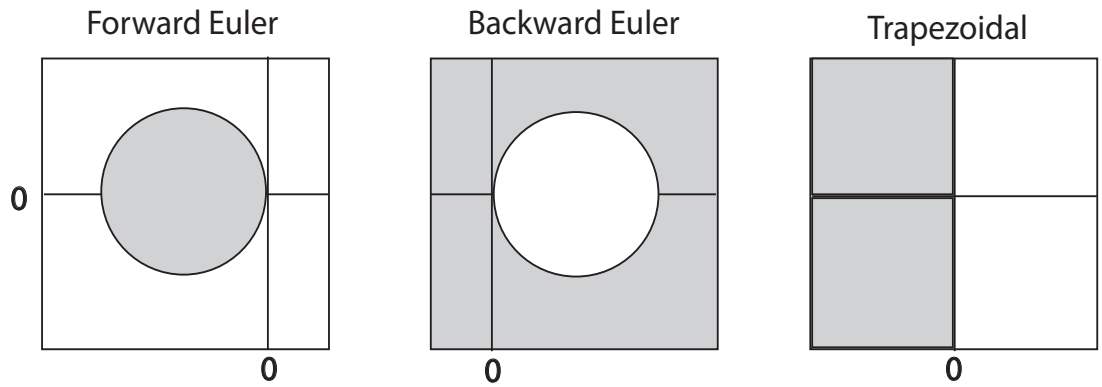
Euler's method applied to this problem gives $U^{n+1} = (1 + \Delta t\lambda)U^n$ and we say this method is absolutely stable when $|1 + \Delta t\lambda| \leq 1$; otherwise it is unstable.

Note that there are two parameters Δt and λ, but only their product $z \equiv \Delta t\lambda$ matters. The method is stable whenever $-2 \leq z \leq 0$, and we say that the *interval of absolute stability* for Euler's method is $[-2, 0]$.

The problem (6) is called an eigenvalue problem.

In the linear case, it is the eigenvalues of the coefficient matrix that are important in determining stability. If the system is nonlinear, then typically linearize it first then consider the eigenvalues of the Jacobian matrix.

[Region of absolute stability] If λ is complex (of course the time step Δt is always a real number)



Stability Regions

5.2 Stability regions for linear multistep methods

Recall the linear multistep methods(LMM) :

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = \Delta t \sum_{j=0}^{r} \beta_j f(U^{n+j}, t_{n+j}). \tag{7}$$

The region of absolute stability is found by applying the method to $u' = \lambda u$, obtaining

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = \Delta t \sum_{j=0}^{r} \beta_j \lambda U^{n+j} \quad \Longrightarrow \quad \sum_{j=0}^{r} (\alpha_j - z\beta_j) U^{n+j} = 0. \tag{8}$$

Note! $z = \Delta t \lambda$ is important not the values of Δt or λ separately.

From (8), we consider a polynomial

$$\pi(\eta; z) := \sum_{j=0}^{r} (\alpha_j - z\beta_j)\eta^j = \rho(\eta) - z\sigma(\eta)$$

and it is called the *stability polynomial*.

[Definition] Region of absolute stability

The region of absolute stability for the LMM is the set of points z in the complex plane for which the polynomial $\pi(\eta; z)$ satisfies the root condition:

$$|\eta_i| \le 1 \quad \text{for } i = 1, 2, \cdots, r.$$

If η_i is a repeated root, then $|\eta_i| < 1$.

[Example] Forward Euler:

$$\pi(\eta; z) = \eta - (1 + z)$$

which has a single root $\eta = 1 + z$. There fore the stability region is

$$|1 + z| \leq 1.$$

[Example] Backward Euler:

$$\pi(\eta; z) = (1 - z)\eta - 1$$

which has a single root $\eta = (1 - z)^{-1}$. There fore the stability region is

$$|1 - z|^{-1} \leq 1 \quad \Longleftrightarrow \quad |1 - z| \geq 1.$$

[Example] Trapezoidal, Leapfrog, $\cdots$

5.3 Plotting stability regions

- **The boundary locus method for LMM**
  Let $z \in \mathbb{C}$ be in the stability region of an LMM, that is

  $$|\eta| \leq 1.$$

  Moreover, if z is on the boundary of the stability region, then $\pi(\eta; z)$ must have at least one root η_j with $|\eta_j| = 1$ thus

  $$\eta_j = e^{i\theta}, \quad \text{for } \theta \in [0, 2\pi].$$

  Then we have

  $$\pi(e^{i\theta}; z) = 0 \Rightarrow \rho(e^{i\theta}) - z\sigma(e^{i\theta}) = 0 \Rightarrow z = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}.$$

  If we know θ, then we can find z. Therefore, since every point z on the boundary of the stability region must be of this form for some value of θ in $[0, 2\pi]$, we can simply plot the parameterized curve

  $$z(\theta) \equiv \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$$

  for $0 \leq \theta \leq 2\pi$ to find the locus of all points which are potentially on the boundary of the stability region.
  $\Rightarrow$ This yields the stability region!!!
  In order to determine which side of this curve is the *interior* of the boundary, check any value z on one side or the other and see if the polynomial satisfies the root condition.

- Plotting stability regions of one-step methods
  If we apply a one-step method to the test problem $u' = \lambda u$, we get

$$U^{n+1} = P(z)U^n,$$

  where $P(z)$ is some function of $z = \lambda \Delta t$.
  If the method is consistent, $P(z) \approx e^z$.
  From the definition of absolute stability, we can simply represent the region of absolute stability for a one-step method is

$$\{z \in \mathbb{C} : |P(z)| \leq 1\} \quad \Longrightarrow \quad |U^n| \leq |P(z)|^n |U^0| \text{ : uniformly bounded in } n$$

  [Example] (Fourth order Runge-Kutta method) applied to $u' = \lambda u'$.

$$P(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4.$$

  - graphical approach : on fine grid of points in complex plane, approximate the level set where function $|P(z)|$ has the value 1.
    $\Longrightarrow$ *contour* command in MATLAB.